Observing Through the Turbulent Atmosphere

Andreas Quirrenbach

(University of California, San Diego)

Convolution, Correlation, and Structure Function

- $g * h \equiv \int_{-\infty}^{\infty} g(t-\tau)h(\tau)\mathrm{d}\tau$
- $\mathrm{Corr}(g,h) \equiv \int_{-\infty}^{\infty} g(t+\tau)h(\tau)\mathrm{d}\tau$
- $D_g(t_1,t_2) \equiv \langle|g(t_1)-g(t_2)|^2\rangle$ **(Structure Function)**

- $g * h \Longleftrightarrow G(f)H(f)$ **(Convolution Theorem)**
- $\mathrm{Corr}(g,h) \Longleftrightarrow G(f)H^*(f)$ **(Correlation Theorem)**
- $\mathrm{Corr}(g,g) \Longleftrightarrow |G(f)|^2$ **(Wiener-Khinchin-Theorem)**
- $\text{Total Power} \equiv \int_{-\infty}^{\infty} |h(t)|^2\mathrm{d}t = \int_{-\infty}^{\infty} |H(f)|^2\mathrm{d}f$ **(Parseval's Theorem)**

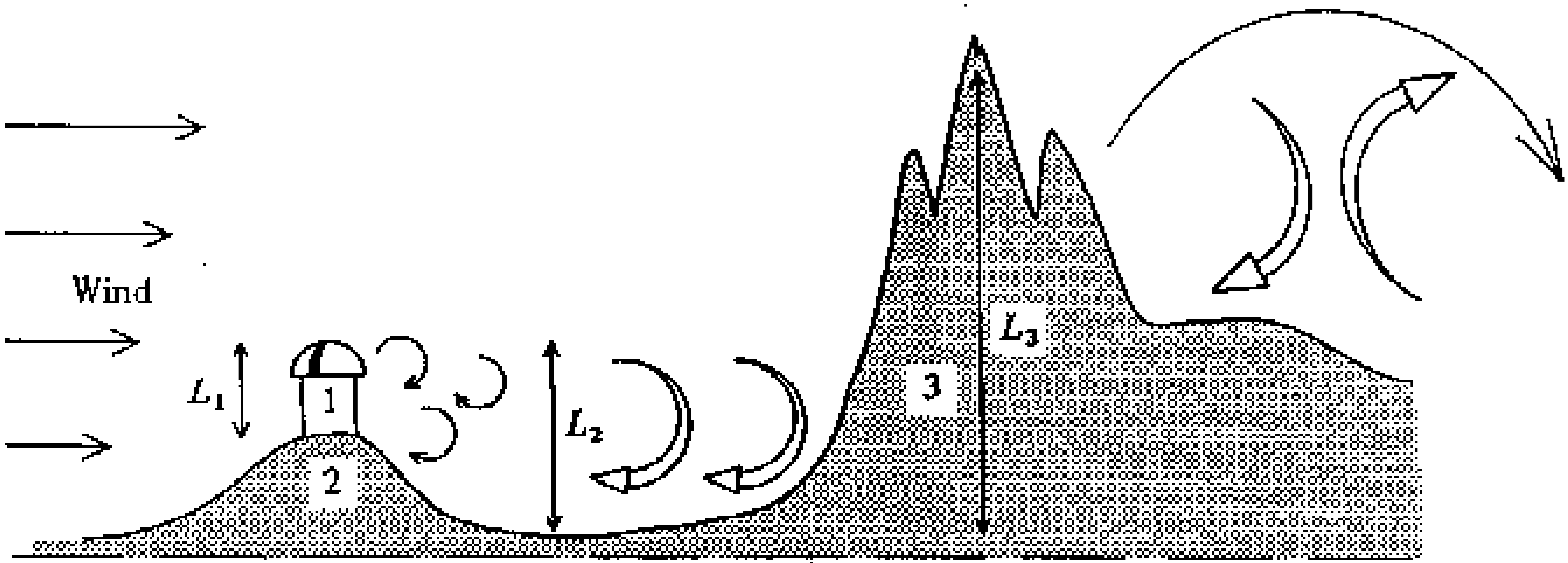


Fig. 2.13. Schematic representation of the generation of turbulence in the atmosphere by different obstacles. The amplitude of the temperature fluctuations depends on the amplitude of the turbulence and on the deviation of the actual temperature gradient from the adiabatic gradient. The scales L_1, L_2, L_3 are characteristic of the external scales of turbulence caused by wind around the obstacles 1, 2 and 3

The Kolmogorov Turbulence Model

- The Reynolds number $Re = VL/\nu$ for atmospheric flows is of order $Re \gtrsim 10^6$, i.e, the atmosphere is highly turbulent.
- Turbulent energy is generated on large scale L_0, dissipated on small scale l_0.
- L_0 is called "outer scale", l_0 "inner scale".
- In the "inertial range" between l_0 and L_0, there is a universal description for the turbulence spectrum.
- The only two relevant parameters are the rate of energy generation ε and the kinematic viscosity ν.

The Structure Function for Kolmogorov Turbulence

- The units of ν are $\mathrm{m^2\,s^{-1}}$, those of ε are $\mathrm{J\,s^{-1}\,kg^{-1} = m^2\,s^{-3}}$.
- The velocity structure function can be written as:

$$\begin{aligned} D_v(R_1, R_2) &\equiv \langle |v(R_1) - v(R_2)|^2 \rangle \\ &= \alpha \cdot f\left(|R_1 - R_2| / \beta\right) \quad . \end{aligned}$$

- The dimensions of α are velocity squared $\Rightarrow \alpha = \nu^{1/2}\varepsilon^{1/2}$.
- The dimensions of β are length $\Rightarrow \beta = \nu^{3/4}\varepsilon^{-1/4}$.
- In the inertial range the dependence on ν must drop out $\Rightarrow$

$$D_v(R_1, R_2) = C_v^2\,|R_1 - R_2|^{2/3} \quad ,$$

where C_v^2 is a constant describing the turbulence strength.

Structure Function and PSD of Refractive Index

- Turbulence carries "parcels" of air with different temperature, and thus with different index of refraction.
- The corresponding structure functions are:

$$D_T(R_1, R_2) = C_T^2 \cdot |R_1 - R_2|^{2/3} \quad , \text{ and}$$
$$D_n(R_1, R_2) = C_N^2 \cdot |R_1 - R_2|^{2/3} \quad ,$$

  with $C_N = (78 \cdot 10^{-6} P[\mathrm{mbar}]/T^2[K]) \cdot C_T$.
- The structure function D is related to the covariance B by:

$$D(R) = 2(B(0) - B(R))$$

- The covariance is the Fourier transform of the power spectral density Φ (Wiener-Khinchin theorem).
- For Kolmogorov turbulence $\Phi(\kappa) \propto \kappa^{-5/3}$.

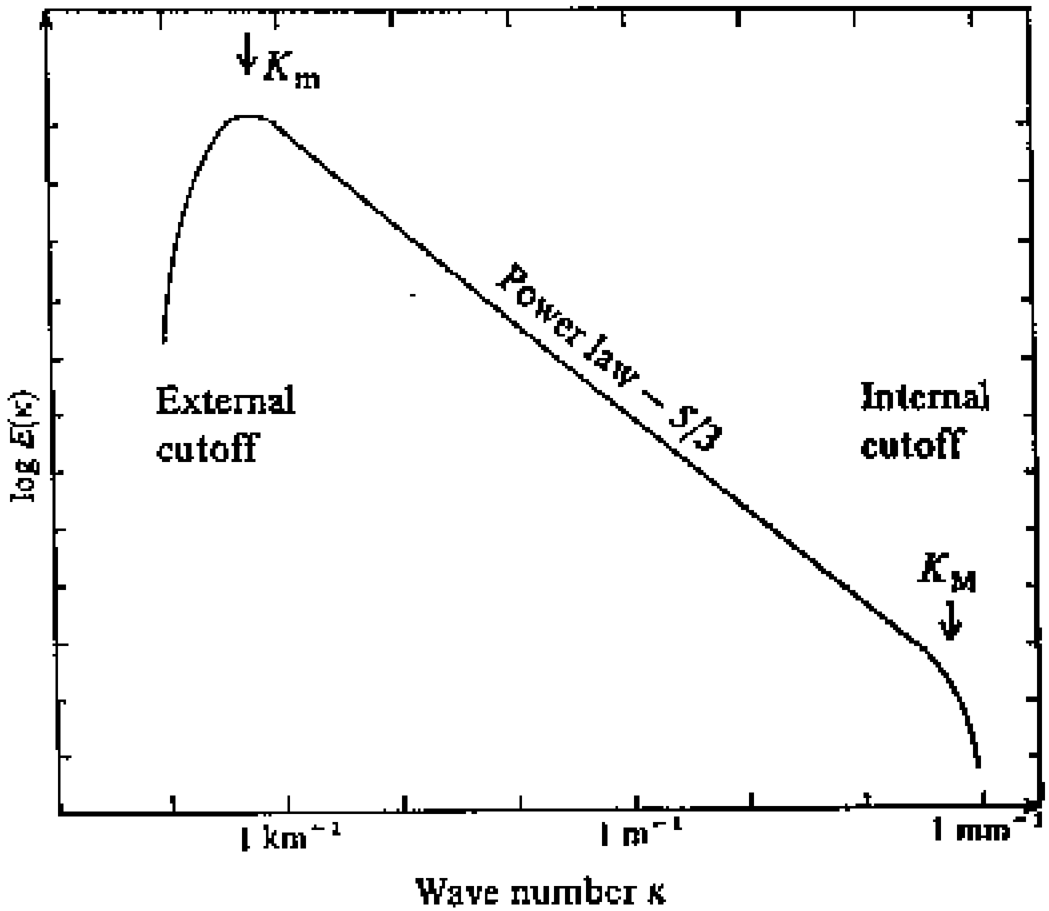


Fig. 2.12. One-dimensional power spectrum $E(\kappa)$ of the velocity fluctuations in a turbulent fluid, where the turbulence is isotropic and fully developed between the two scales L_0 and l_0 (turbulence obeying Kolmogorov's law in this interval). The corresponding wave numbers are $\kappa_m = 1/L_0$ and $\kappa_M = 1/l_0$. The ordinate is $\log E(\kappa)$. A variation in intensity of the turbulence (or of the energy injected at the scale L_0) results in a vertical shift of the curve

Effects of Turbulent Layers

We look at the propagation of a wavefront $\psi(x) = \exp i\phi(x)$ through a turbulent layer of thickness δh at height h. The phase shift produced by refractive index fluctuations is

$$\phi(x) = k \int_h^{h+\delta h} dz\, n(x,z) \quad ,$$

where $k = 2\pi/\lambda$.

The *coherence function* of the wavefront is:

$$\begin{aligned} B_h(r) &\equiv \langle \psi(x+r)\psi^*(x) \rangle \\ &= \langle \exp i\left[\phi(x) - \phi(x+r)\right] \rangle \\ &= \exp -\tfrac{1}{2}\langle \left|\phi(x) - \phi(x+r)\right|^2 \rangle \\ &= \exp -\tfrac{1}{2} D_\phi(r) \end{aligned}$$

We have to calculate $D_\phi(r)$.

Calculation of the Phase Structure Function

For δh much larger than the correlation scale of the fluctuations, the *covariance* of ϕ can be written as:

$$
\begin{aligned}
B_\phi(r) &\equiv \langle \phi(x)\phi(x+r)\rangle \\
&= k^2 \delta h \int_{-\infty}^{\infty} dz\, B_N(r,z) \quad \Longrightarrow
\end{aligned}
$$

$$
\begin{aligned}
D_\phi(r) &= 2(B_\phi(0) - B_\phi(r)) \\
&= 2k^2 \delta h \int_{-\infty}^{\infty} dz\, (B_N(0,z) - B_N(r,z)) \\
&= k^2 \delta h \int_{-\infty}^{\infty} dz\, (D_N(r,z) - D_N(0,z)) \\
&= k^2 \delta h\, C_N^2 \int_{-\infty}^{\infty} dz\, \left[\left(r^2+z^2\right)^{1/3} - z^{2/3}\right] \\
&= \frac{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{6})}{5\Gamma(\frac{2}{3})}\, k^2 \delta h\, C_N^2\, r^{5/3} \\
&= 2.914\, k^2 \delta h\, C_N^2\, r^{5/3}
\end{aligned}
$$

Phase Coherence Function and Fried Parameter

The phase coherence function for a turbulent layer is now:

$$B_h(r) = \exp\left[-\tfrac{1}{2}\left(2.914\, k^2 C_N^2\, \delta h\, r^{5/3}\right)\right]$$

Integration over the whole atmosphere, and taking into account the zenith angle z, gives:

$$B(r) = \exp\left[-\tfrac{1}{2}\left(2.914\, k^2 (\sec z) r^{5/3} \int dh\, C_N^2(h)\right)\right]$$

We now define the *Fried parameter* r_0 by:

$$r_0 \equiv \left[0.423\, k^2 (\sec z) \int dh\, C_N^2(h)\right]^{-3/5}$$

and can write

$$B(r) = \exp\left[-3.44 \left(\frac{r}{r_0}\right)^{5/3}\right] \quad , \quad D(r) = 6.88 \left(\frac{r}{r_0}\right)^{5/3} \quad .$$

Optical Image Formation

- The complex amplitude A of a wave ψ diffracted at an aperture P with area Π is given by Huygens' principle:

$$A(\alpha) = \frac{1}{\sqrt{\Pi}} \int \psi(x) P(x) \exp(-2\pi i \alpha x/\lambda) dx$$

- With $u \equiv x/\lambda$:

$$A(\alpha) = \frac{1}{\sqrt{\Pi}} FT[\psi(u) P(u)]$$

- The illumination in the focal plane ("point spread function") is:

$$S(\alpha) = |A(\alpha)|^2 = \frac{1}{\Pi} |FT[\psi(u) P(u)]|^2$$

- Autocorrelation theorem:

$$\langle S(f) \rangle = B(f) \cdot T(f) \quad \text{with} \quad T(f) = \frac{1}{\Pi} \int P(u) P^*(u+f) du$$

Diffraction-Limited and Seeing-Limited Images

- Definition of resolving power R of an optical system:

$$R = \int B(f)T(f)df$$

- In the absence of turbulence, $B \equiv 1$, and

$$\begin{aligned} R_{\mathrm{tel}} &= \int T(f)df = \frac{1}{\Pi}\int\int P(u)P^*(u+f)du df \\ &= \frac{1}{\Pi}\left|\int P(u)du\right|^2 = \frac{\pi}{4}\left(\frac{D}{\lambda}\right)^2 \end{aligned}$$

- For strong turbulence, $T = 1$ in the region where B is non-zero, and

$$\begin{aligned} R_{\mathrm{atm}} &= \int B(f)df = \int \exp\left(-Kf^{5/3}\right)df \\ &= \frac{6\pi}{5}\Gamma\left(\tfrac{6}{5}\right)K^{-6/5} = \frac{6\pi}{5}\Gamma\left(\tfrac{6}{5}\right)\left(3.44\left(\tfrac{\lambda}{r_0}\right)^{5/3}\right)^{-6/5} = \frac{\pi}{4}\left(\frac{r_0}{\lambda}\right)^2 \end{aligned}$$

Significance of the Fried Parameter r_0

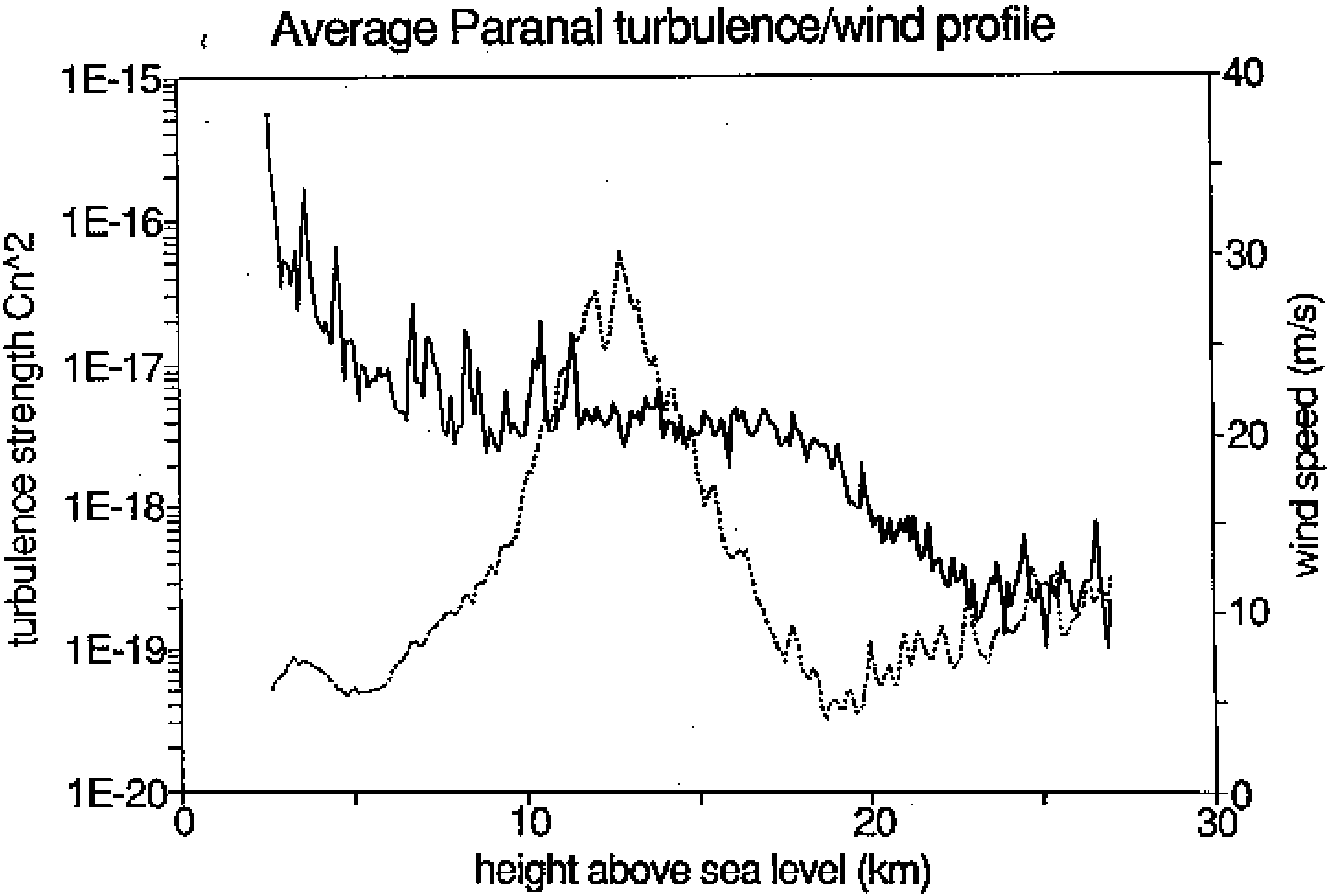
- The effective resolution of long exposures through the atmosphere is the same as the resolution obtained with a telescope of diameter r_0.
- The phase variance over an aperture with diameter r_0 is approximately $1\,\mathrm{rad}^2$.
- r_0 depends on the turbulence profile $C_N^2(h)$, the zenith angle z, and the observing wavelength λ.
- The wavelength dependence is $r_0 \propto \lambda^{6/5}$; this leads to an image size ("seeing") $\alpha \propto \lambda/r_0 \propto \lambda^{-1/5}$.
- At good sites, typical values for r_0 at $\lambda = 500\,\mathrm{nm}$ are in the range $10\ldots20\,\mathrm{cm}$; this corresponds to $\alpha = 0.5'' \ldots 1''$.

The Strehl Ratio

- The quality of an imaging system is often measured by the *Strehl ratio* S, defined as the peak intensity in the image divided by the peak intensity in a diffraction-limited image.
- For Gaussian fluctuations, $S = \exp(-\sigma_\phi^2)$.
- The Hubble Space Telescope has $S \approx 0.1$ (without corrective optics).
- A telescope with diameter r_0 gives $S = 0.37$.
- If $S \gtrsim 0.1$, image deconvolution software can usually be used to obtain diffraction-limited images, but the dynamic range is limited.

The Taylor "Frozen Turbulence" Hypothesis and τ_0

- The time constant for changes in the turbulence pattern is usually much longer than the time it takes the wind to blow the turbulence past the telescope aperture.
- Atmospheric turbulence is often dominated by a single layer.
- The temporal behavior of the turbulence can therefore be characterized by a time constant $\tau_0 \equiv r_0/v$, where v is the wind velocity in the dominant layer.
- Observations with exposure time $t << \tau_0$ (so-called "short exposures") produce images through one instantaneous realization of the atmosphere ("speckle images"); observations with $t >> \tau_0$ average over the atmospheric random process.



Average Paranal turbulence/wind profile
1E-15
1E-16
1E-17
1E-18
1E-19
1E-20
turbulence strength Cn^2
40
30
20
10
0
wind speed (m/s)
0
10
20
30
height above sea level (km)

Anisoplanatism

- The light from two stars separated by an angle θ passes through different patches of the atmosphere and therefore experiences different phase variations.
- It can be shown that the variance of the phase difference between the two stars is given by:

  $$\langle\sigma_\theta^2\rangle = \left(\frac{\theta}{\theta_0}\right)^{5/3}$$

- In this relation, the *isoplanatic angle* is given by:

  $$\theta_0 \equiv \left[2.914k^2(\sec z)^{8/3}\int dh\, C_N^2(h)h^{5/3}\right]^{-3/5}$$

- Note: the short-exposure point spread functions for two stars separated by more than θ_0 are different, but the long-exposure psf's are (nearly) identical.
- Anisoplanatism is dominated by high-altitude turbulence.

Scintillation

- The geometric optics approximation of propagation is only valid for propagation paths shorter than the *Fresnel length* $d = r_0^2/\lambda$.
- For $r_0 = 10\,\mathrm{cm}$, $\lambda = 500\,\mathrm{nm}$, the Fresnel length is $20\,\mathrm{km}$. The geometric approximation is therefore a good first-order approach, but diffraction is not negligible, especially at short wavelengths, large zenith angles, and poor observing sites.
- Diffraction gives rise to *scintillation*, i.e., intensity fluctuations that are important for photometry if the exposure time is short.
- Scintillation is an interference phenomenon, and therefore highly chromatic.
- Scintillation is dominated by high-altitude turbulence.

Table 4.6. The first few Zernike polynomials and corresponding optical aberrations

Radial	Azimuthal frequency m			
degree n	$m=0$	$m=1$	$m=2$	$m=3$
0	$Z_1 = 1$ Piston			
1		$Z_2 = 2r\cos\alpha$ $Z_3 = 2r\sin\alpha$ Tip-tilt		
2	$Z_4 = \sqrt{3}$ $(2r^2-1)$ Defocus		$Z_5 = \sqrt{6}r^2\sin 2\alpha$ $Z_6 = \sqrt{6}r^2\cos 2\alpha$ Astigmatism	
3		$Z_7 = \sqrt{8}(3r^3$ $-2r)\sin\alpha$ $Z_8 = \sqrt{8}(3r^3$ $-2r)\cos\alpha$ Coma		$Z_9 = \sqrt{8}r^3$ $\times\sin 3\alpha$ $Z_{10} = \sqrt{8}r^3$ $\times\cos 3\alpha$
4	$Z_{11} = \sqrt{5}$ $(6r^4-6r^2+1)$ Spherical aberration		$Z_{12} = \sqrt{10}$ $(4r^4-3r^2)\cos 2\alpha$ $Z_{13} = \sqrt{10}$ $(4r^4-3r^2)\sin 2\alpha$	

Table 4.6 gives the classification, the formula and the equivalent aberrations in classical optics for the first few Zernike polynomials. This basis is orthogonal on a circular pupil

TABLE IV. Zernike-Kolmogoroff residual errors (Δ_J). (D is the aperture diameter.)

$\Delta_1 = 1.0299\ (D/r_0)^{5/3}$	$\Delta_{12} = 0.0352\ (D/r_0)^{5/3}$
$\Delta_2 = 0.582\ (D/r_0)^{5/3}$	$\Delta_{13} = 0.0328\ (D/r_0)^{5/3}$
$\Delta_3 = 0.134\ (D/r_0)^{5/3}$	$\Delta_{14} = 0.0304\ (D/r_0)^{5/3}$
$\Delta_4 = 0.111\ (D/r_0)^{5/3}$	$\Delta_{15} = 0.0279\ (D/r_0)^{5/3}$
$\Delta_5 = 0.0880\ (D/r_0)^{5/3}$	$\Delta_{16} = 0.0267\ (D/r_0)^{5/3}$
$\Delta_6 = 0.0648\ (D/r_0)^{5/3}$	$\Delta_{17} = 0.0255\ (D/r_0)^{5/3}$
$\Delta_7 = 0.0587\ (D/r_0)^{5/3}$	$\Delta_{18} = 0.0243\ (D/r_0)^{5/3}$
$\Delta_8 = 0.0525\ (D/r_0)^{5/3}$	$\Delta_{19} = 0.0232\ (D/r_0)^{5/3}$
$\Delta_9 = 0.0463\ (D/r_0)^{5/3}$	$\Delta_{20} = 0.0220\ (D/r_0)^{5/3}$
$\Delta_{10} = 0.0401\ (D/r_0)^{5/3}$	$\Delta_{21} = 0.0208\ (D/r_0)^{5/3}$
$\Delta_{11} = 0.0377\ (D/r_0)^{5/3}$	

$$\Delta_J \sim 0.2944\, J^{-\sqrt{3}/2}\ (D/r_0)^{5/3} \quad \text{(For large } J\text{)}$$